Comp. Appl. Math.
DOI 10.1007/s40314-016-0382-0

The influence of interval arithmetic on the shape of uncertainly defined domains modelled by closed curves

Eugeniusz Zieniuk[1] · **Andrzej Kużelewski**[1] · **Marta Kapturczak**[1]

Received: 3 December 2015 / Revised: 3 August 2016 / Accepted: 1 September 2016

Abstract The paper presents an unambiguous modelling of imprecisely defined shapes of closed curves using classical and directed interval arithmetic. The authors focus on the development of an effective strategy of modelling interval smooth closed curves (which enforce C^2 continuity in points at which adjacent interval segments join) using interval cubic Bézier segments. For this purpose, algebraic relationships between Bézier and de Boor control points, formerly known for precisely defined curves, are generalized. We obtain interval control points that define interval closed curves. Additionally, the reliability of such way of modelling of closed curves is examined. We directly apply classical and directed interval arithmetic to mentioned relationships and try to solve obtained interval systems of algebraic equations. However, we obtain ambiguous solutions. Therefore, we propose our new strategy of modification of directed interval arithmetic to obtain reliable and unambiguous shapes of interval closed curves.

Keywords Interval Bézier curve · Interval B-spline curve · Directed interval arithmetic · Uncertainty · Boundary modelling · Boundary value problems

Mathematics Subject Classification 65G30 · 65G40 · 65D17 · 68U07

Communicated by Luz de Teresa.

✉ Andrzej Kużelewski
akuzel@ii.uwb.edu.pl

Eugeniusz Zieniuk
ezieniuk@ii.uwb.edu.pl

Marta Kapturczak
mkapturczak@ii.uwb.edu.pl

1 Faculty of Mathematics and Informatics, University of Bialystok, Konstantego Ciołkowskiego 1M, 15-245 Białystok, Poland

Published online: 15 September 2016

1 Introduction

One of the most popular way to create various kinds of curves in computer graphics is the use of Bézier (Galvez and Iglesias 2013; Hanniel et al. 2009) or B-spline (Csebfalvi 2010; Gonzalez-Hidalgo et al. 2013) curves. In the age of computer technology this approach became very effective. It is known that an accurate representation of the curve on the screen of any computer requires the use of mathematical formalism. However, mathematical defining of curves combined with computer technology offers many new possibilities in modelling various kinds of shapes. For example, it allows to change in effective way the shape of modelled boundary during searching its optimal shape (Hao and Sun 2013; Zieniuk et al. 2007).

One of the main advantages of application of mentioned curves is easiness of its shape modification. From mathematical point of view every curve is a function. A curve is defined by control points which simplify the process of its shape modification. These points are usually named by the name of the inventor of the curve, for example Bézier or de Boor control points. The curves have different advantages and disadvantages, therefore the way of using curves in practice is strictly connected with their particular application. In the former studies the authors used the curves to model the shape of boundary value problems by the method of parametric integral equations systems (PIES) (Bołtuć and Zieniuk 2011; Kużelewski and Zieniuk 2015a, b; Zieniuk and Szerszeń 2014; Zieniuk et al. 2013).

Classical (precise) curves are modelled by precisely defined segments. However, in the literature one can find interval generalizations of algebraic (Martin et al. 2002), rational (Chen and Deng 2004), Bézier (Lin et al. 2002; Sederberg and Farouki 1992) or B-spline (Tuohy et al. 1997) curves. Interval curves differ from classical ones in that the real numbers representing control point coordinates are replaced by classical interval numbers. Interval definition of control points is more realistic than precise one, because in practice we are not able to define them in an accurate way. It is well-known that all measurements are affected by measurement errors. These errors can be mathematically defined by, among others, interval analysis (Moore 1966; Hansen and Walster 2004; Markov 1995). In the theoretical considerations the use of intervals to define such errors seems to be very reasonable. However, further inclusion of interval numbers into mathematical models and their application during numerical computations can be very troublesome. First of all, there is the problem of overestimation of the width of intervals of final solutions (Hansen 1997; Stahl 1997). In this case, application of interval numbers may become useless from practical point of view.

Former studies related to interval Bézier or B-spline curves are limited to consideration of the individual segments (Lin et al. 2002; Chen and Lou 2000; Ismail 2014a, b). The authors of these papers try to reduce as well as elevate the degree of interval Bézier curves. The application of the curves in modelling boundary value problems by interval boundary element method is presented in Piasecka-Belkhayat (2011) and Zalewski and Mullen (2009) as well as by interval finite element method in Muhanna et al. (2007). However, the shape of the domain was defined by precise curves only. The authors of all mentioned papers used classical interval arithmetic proposed in the mid-1960s of the last century by Moore (1966). The biggest flaw of this arithmetic is mentioned overestimation of width of intervals. In this paper the authors decided to examine and apply recent developments in interval analysis (for example directed interval numbers), mainly in the context of interval modelling of the shape of closed curves. These curves are widely used in practice, however in our studies we are interested in using them for unambiguous modelling of the shape of boundary in uncertainly defined boundary value problems. In order to solve these problems we can apply, among others, PIES method which was widely tested for precisely defined problems (Bołtuć and Zieniuk 2011; Kużelewski and Zieniuk 2015a; Zieniuk and Szerszeń 2014; Zieniuk 2003).

Generalization of PIES method on uncertainly defined boundary problems requires to develop an effective strategy of unambiguous modelling of uncertainly defined shape of boundary. In this paper, we use interval cubic Bézier segments patched together producing interval closed curve (sometimes called composite Bézier curve) for efficient modelling of uncertainly defined shape of boundary. However, to enforce effectively C^2 continuity in point at which adjacent interval segments join, interval Bézier segments have to be mapped to interval B-spline segments. For this purpose we have to solve interval system of algebraic equations using interval arithmetic. In the literature we can find two main kinds of interval arithmetic: classical (Moore 1966; Moore et al. 2009) and directed (Markov 1995; Dimitrova et al. 1992).

Therefore, the authors pose the following questions:

1. Will direct application of classical or directed interval arithmetic to create interval linear Bézier segments patched together producing interval closed curve give correct and unambiguous solutions?
2. Which interval arithmetic, classical or directed, should be applied to accurately solve interval systems of algebraic equations necessary to obtain smooth interval closed curves (which enforce C^2 continuity in points at which adjacent interval segments join)?
3. Are obtained interval shapes of closed curves the same as expected ones from the point of view of interval modelling of the shape of boundary?
4. Are interval curves obtained on the basis of proposed concepts accurate and suitable for modelling interval domains in boundary value problems?

The main goal of this paper is to find answers to posed questions. Our researches should allow to apply the strategy proposed in this paper to model and solve uncertainly defined boundary problems. Nevertheless, the conclusions of the study are more general, because they are related to the strategy of unambiguous modelling of interval smooth closed curves (enforced C^2 continuity in point at which adjacent interval segments join) regardless of their practical application.

The tests show that uncertain defining of the shape of boundary requires, at first, selection of appropriate interval arithmetic that should be used during numerical computations. Furthermore, in order to obtain unambiguous and reliable interval shape of boundary in all quadrants of Cartesian coordinate system, we propose the modification of directed interval arithmetic. An additional goal of this paper is to perform visual analysis of the shape of interval curves in order to verify their usefulness in practical modelling of interval boundary problems.

The summary of classical and directed interval arithmetic is presented in Sect. 2 whereas Gauss elimination method adopted to solve interval systems of algebraic equations in Sect. 3. Section 4 contains the concept of modelling of interval closed curves. In Sect. 5 we describe analysis of the influence of interval arithmetic on the accurateness of modelling of interval curves. In Sect. 6 we present new modification of directed interval arithmetic addressed to obtain unambiguous and reliable shape of interval closed curves. At other sections we describe the tests of application of modified directed interval arithmetic referred to more sophisticated examples.

2 Interval arithmetic

Modelling of boundary value problems with uncertainly defined shape of boundary requires the use of interval curves. We apply interval numbers to describe such problems, and interval arithmetic (Moore 1966) to solve them.

Classical interval number $\mathbf{x}$ (or shorter interval) is the set of real numbers x which meet the condition (Moore 1966; Moore et al. 2009):

$$\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\} \tag{1}$$

where $\underline{x}$-is infimum and $\overline{x}$-is supremum of interval $\mathbf{x}$. This number is also called proper number.

In order to operate on these numbers classical interval arithmetic was developed (Moore 1966) and it is generally defined by the formula (Hayes 2003):

$$\mathbf{x} \circ \mathbf{y} = [\underline{x}, \overline{x}] \circ [\underline{y}, \overline{y}] = [\min(\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y}), \max(\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y})] \tag{2}$$

where $\circ \in \{+, -, \cdot, /\}$ (in case of division "/": $0 \notin \mathbf{y}$).

The disadvantage of classical interval numbers is the lack of inverse and opposite elements. As the result of subtraction we obtain $0 \in \mathbf{x} - \mathbf{x}$, while division $1 \in \mathbf{x}/\mathbf{x}$. It results in the problem with solving even the simplest equation as well as the systems of linear equations. Each subtract or divide operation may increase the width of possible solutions (overestimation problem).

In order to avoid mentioned flaws of classical interval arithmetic we decide to apply extended (directed) interval arithmetic and directed interval numbers (Markov 1995; Dimitrova et al. 1992). Directed interval number (or shorter directed interval) $\mathbf{x}$ is the set of ordered real numbers x described by Markov (1995):

$$\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in \mathbf{x} \mid \underline{x}, \overline{x} \in \mathbb{R}\} \tag{3}$$

Directed intervals are the extension of the set of classical interval numbers. They allow to define improper intervals whose right endpoint is smaller than left one ($\overline{x} \leq x \leq \underline{x}$). In addition, two arithmetic operators of subtraction $\ominus$ and division $\oslash$ were proposed in Markov (1995):

$$\mathbf{x} \ominus \mathbf{y} = [\underline{x} - \underline{y}, \overline{x} - \overline{y}] \tag{4}$$

$$\mathbf{x} \oslash \mathbf{y} = \begin{cases} [\underline{x}/\underline{y}, \overline{x}/\overline{y}] & \text{for } \mathbf{x} > 0, \mathbf{y} > 0 \\ [\overline{x}/\overline{y}, \underline{x}/\underline{y}] & \text{for } \mathbf{x} < 0, \mathbf{y} < 0 \\ [\overline{x}/\underline{y}, \underline{x}/\overline{y}] & \text{for } \mathbf{x} > 0, \mathbf{y} < 0 \\ [\underline{x}/\overline{y}, \overline{x}/\underline{y}] & \text{for } \mathbf{x} < 0, \mathbf{y} > 0 \\ [\overline{x}/\underline{y}, \underline{x}/\underline{y}] & \text{for } \mathbf{x} \ni 0, \mathbf{y} < 0 \\ [\underline{x}/\overline{y}, \overline{x}/\overline{y}] & \text{for } \mathbf{x} \ni 0, \mathbf{y} > 0 \end{cases} \tag{5}$$

Hence, we can obtain opposite element ($0 = \mathbf{x} \ominus \mathbf{x}$) and inverse one ($1 = \mathbf{x} \oslash \mathbf{x}$).

3 Interval system of algebraic equations

General matrix form of interval system of algebraic equations can be presented as follows (Neumaier 1990; Shary 2012):

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{6}$$

where $\mathbf{A}$ is a matrix of interval coefficients, $\mathbf{x}$ is a vector of interval solutions, whereas $\mathbf{b}$ is a vector of intervals. There are many methods of solving such system of equations. Due to the way of defining the coordinates of control points we should use directed interval numbers. Therefore, the choice of method of solving the interval system of equations is strictly connected with directed interval arithmetic.

The interval methods appearing in the literature include Jacobi iterative method described by Markov (1999); Zieniuk et al. (2016), LU decomposition method (Goldsztejn and Chambert 2007) and the Gauss elimination method (Neumaier 1990; Zieniuk et al. 2016), which is the most commonly used method. During the tests we decided to present the results of the Gauss elimination method only. It is connected with the fact that LU decomposition method returns similar results and Jacobi iterative method has a problem with convergence in many cases.

The best-known and widely used method in the real numbers as well as interval domain is the Gauss elimination method. A major step in solving interval system of equations using this method is to transform the matrix coefficients from the formula (6) into the following form (Neumaier 1990):

$$\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \dots & \mathbf{a}_{1n} \\ 0 & \mathbf{a}_{22} & \dots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{a}_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} \tag{7}$$

To achieve the matrix **A** and the vector **b** the following formulas are used:

$$\mathbf{a}_{jk}^{(i+1)} = \mathbf{a}_{jk}^{(i)} \ominus \mathbf{a}_{ik}^{(i)} \cdot \mathbf{a}_{ji}^{(i)} \oslash \mathbf{a}_{ii}^{(i)}, \qquad \mathbf{b}_{k}^{(i+1)} = \mathbf{b}_{k}^{(i)} \ominus \mathbf{b}_{i}^{(i)} \cdot \mathbf{a}_{ki}^{(i)} \oslash \mathbf{a}_{ii}^{(i)} \tag{8}$$

where $i = 1, 2, \dots, n$ and $j, k = i + 1, i + 2, \dots, n$

With an upper triangular matrix (7) we can easily determine the value of the vector of solutions. Starting from the last row, elements of the matrix **A** are compared with the value in corresponding row of the vector **b**. The only difference between the presented method and the method known from the real numbers domain is application of directed interval arithmetic.

4 The concept of modelling of interval closed curves

The essence of modelling of uncertainly defined shape of boundary by method PIES is the application of parametric curves widely used in computer graphics, particularly interval Bézier curves of any degree (Sederberg and Farouki 1992):

$$\mathbf{S}_k(s) = \sum_{i=0}^{n} \mathbf{P}_i \mathrm{B}_{i,n}(s), \quad 0 \le s \le 1 \tag{9}$$

where $\mathbf{P}_i$-interval control point, n-degree of curve, $\mathrm{B}_{i,n}$-Bernstein basis functions (Zieniuk 2003) of n-th degree:

$$\mathrm{B}_{i,n}(s) = \binom{n}{i} (1 - s)^{n-i} s^i \tag{10}$$

Interval Bézier curve differs from the classical one that the coordinates of its control points are defined by interval numbers. This is equivalent to define interval Bézier control points as rectangles (presented in Fig. 1a, b).

PIES usually uses two types of Bézier curves: the first or the third degree. Interval curve of the first degree (Fig. 1a) is a parametric linear function used to define the polygonal domains (Zieniuk and Kużelewski 2015). Any segment of the boundary is defined using two

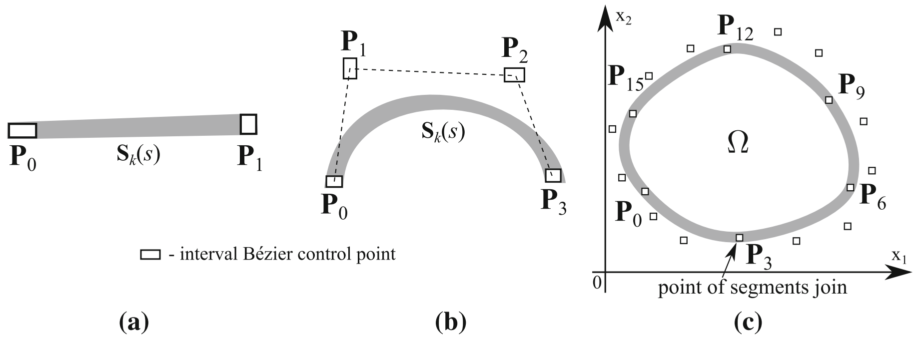


Fig. 1 Interval Bézier curve **a** of first degree (*linear*), **b** of third degree (*cubic*), **c** direct modelling of the boundary by cubic interval Bézier curves

interpolating interval Bézier points, which are called corner points (Zieniuk and Kużelewski 2015).

PIES requires application of interval Bézier curves of the third degree (Fig. 1b) for modelling of smooth uncertainly defined shapes of boundary. In this paper, we consider the use of interval cubic Bézier segments patched together producing interval closed curve (Fig. 1c). The interval closed curve is defined by a small number of interval Bézier control points $\mathbf{P}_0$, $\mathbf{P}_3, \mathbf{P}_6, \mathbf{P}_9, \mathbf{P}_{12}, \mathbf{P}_{15}$ at which adjacent segments join (the points lying on the curve). Defining only these interval control points allows to easier physical modelling of the closed curve in interval way. However, in order to enforce C^2 continuity these points should be mapped into interval de Boor control points on the basis of appropriate algebraic relationships. More detailed studies are presented in Sect. 7.

However, at the beginning of our researches, we need to study the direct use of classical and directed interval arithmetic in numerical computations related to modelling of interval closed curve and draw the appropriate conclusions. These conclusions will be used to apply in further computations the most appropriate interval arithmetic. For this purpose, we consider example of interval closed polygonal chain (polygonal domain) composed of four interval linear segments.

5 Analysis of the influence of interval arithmetic on the interpolation of linear segments

We consider square area domain defined in first quadrant of Cartesian coordinate system by four interval corner points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ (presented in Fig. 2). Firstly, we draw the domain intuitively without any interpolation using interval arithmetic. Next, we apply linear functions and interval arithmetic to define line segments between all corner points (presented as two dots joined by black line). In order to obtain values of coefficients of these functions we create interval system of two equations for each segment. We expect to obtain edges of interval boundary which exactly match the ones from Fig. 2.

To solve such systems, we use interval Gauss elimination method, discussed in Sect. 3. The domain is defined in different quadrants as well as in the centre of the coordinate system. We want to check whether the interpolation for the same domain will be the same in different quadrants due to the use of interval arithmetic. Application of classical interval arithmetic gives senseless solutions for presented shape of boundary. It is difficult to see any relationships between applied control points and obtained line segments, as is shown in Fig. 3a. Surprisingly

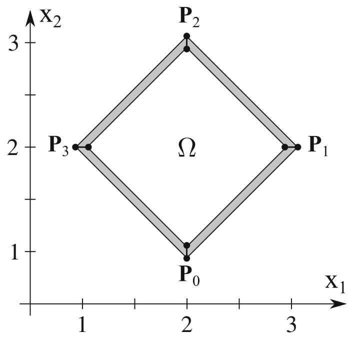


Fig. 2 Expected shape of interval polygonal boundary

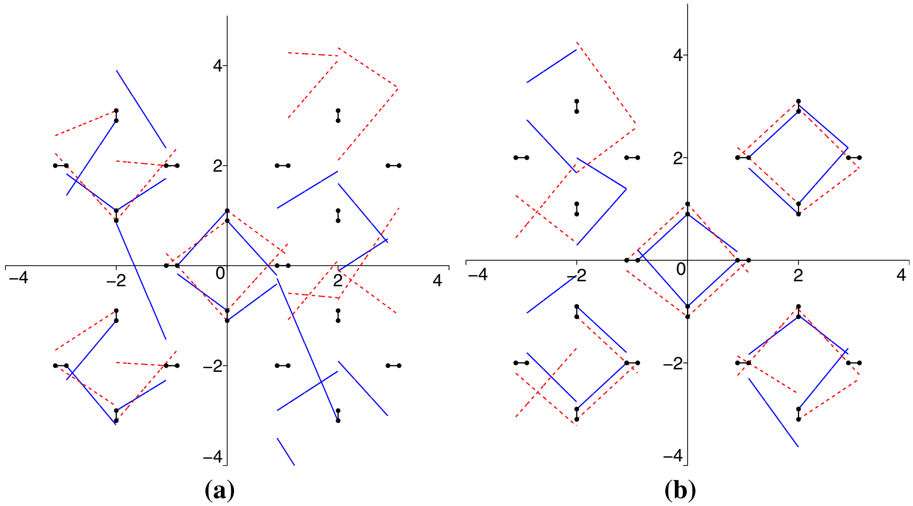


Fig. 3 The shape of interval polygonal boundary interpolated using Gauss elimination and: **a** classical interval arithmetic, **b** directed interval arithmetic

good solution is obtained in the centre of coordinate system, however it is connected with predefined upper triangular matrix.

In contrast, application of directed interval arithmetic and interval Gauss elimination method gives coefficients of linear functions, which allow to draw intersecting line segments (presented in Fig. 3a). Solid red and dotted blue lines are internal and external edge of boundary, respectively, while short black lines present the real position of interval corner points. However, we obtain segments intersect each other outside interval corner points even in the first quadrant of the coordinate system. It does not provide continuity in interval corner points. Furthermore, we obtain different shapes of boundary in all quadrants of the coordinate system. It means that the shape of these segments is dependent on the quadrant of the coordinate system where interval corner points are defined. Moreover, we obtain the shapes of boundary (Fig. 3) significantly differ from expected one (Fig. 2), which we want further apply as the domain of boundary value problem. These simple studies show the shapes of line segments physically obtained by the direct application of interval arithmetic and how they differ from expected ones. That way of modelling of boundary value problems is unacceptable. Interval Gauss elimination method used to solve even small system of linear equations gives solutions that are not allowed to create the same shape of domains in all quadrants of coordinate system.

On the basis of previously gained experience from application of interval arithmetic, we can see, that to obtain unambiguous shape of boundary regardless of the quadrant of coordinate system neither directed nor classical interval arithmetic should be directly used to correctly interpolate boundary (as presented in Fig. 3). Therefore, our further studies are connected with some changes of directed interval arithmetic to obtain ability of correct definition of boundary at any quadrant of coordinate system.

6 The strategy of modification of directed interval arithmetic

On the basis of Fig. 3a, b, we can note that interpolation of the shape of polygonal boundary using direct application of classical as well as directed interval arithmetic results in an incorrect interpolation of interval shape of boundary in both cases. Appropriate ends of obtained line segments do not join interval corner points. Furthermore, we obtain a variety of shapes of boundary with regard to the quadrant of coordinate system where interval corner points are defined. Such way of defining of boundary problems is ambiguous and unacceptable.

However, we should noted that directed interval arithmetic (Fig. 3b) gives two shapes close to the expected shape of boundary. First is defined in the first quadrant, while second is in the centre of coordinate system. Thus, in order to obtain that appropriate ends of segments join interval corner points we propose to map directed interval arithmetic into the positive semi-axis, regardless of coordinates of corner points in coordinate system.

Such mapping may be presented in the following form:

$$\mathbf{x}\cdot\mathbf{y}=\begin{cases}\mathbf{x_s}\cdot\mathbf{y_s}-\mathbf{x_s}\cdot y_m-x_m\cdot\mathbf{y_s}+x_m\cdot y_m & \text{for } \mathbf{x}\le 0, \mathbf{y}\le 0\\ \mathbf{x_s}\cdot\mathbf{y}-x_m\cdot\mathbf{y} & \text{for } \mathbf{x}\le 0, \mathbf{y}>0\\ \mathbf{x}\cdot\mathbf{y_s}-\mathbf{x}\cdot y_m & \text{for } \mathbf{x}>0, \mathbf{y}\le 0\\ \mathbf{x}\cdot\mathbf{y} & \text{for } \mathbf{x}>0, \mathbf{y}>0\end{cases} \tag{11}$$

where for each interval $\mathbf{a}=[\underline{a},\overline{a}]$ we define $\mathbf{a_s}=\mathbf{a}+a_m$, $\mathbf{a}>0$ means $\underline{a}>0$ and $\overline{a}>0$, while $\mathbf{a}\le 0$ means $\underline{a}\le 0$ or $\overline{a}\le 0$ and $a_m=\begin{cases}|\overline{a}| & \text{for } \overline{a}>\underline{a}\\ |\underline{a}| & \text{for } \overline{a}<\underline{a}\end{cases}$. The multiplication $(\cdot)$ is the multiplication of the directed interval arithmetic.

In order to check how proposed mapping of directed interval arithmetic into the positive semi-axis (11) influences the shape of boundary we decide to present solutions on exemplary interval segment. Therefore, we interpolate interval linear segment between two interval points. We obtain interval system of equations that is solved using interval Gauss elimination method. The results are presented in Fig. 4. The captions in legend mean solutions obtained using: “c”-classical interval arithmetic, “d”-directed interval arithmetic and “psa”-mapping directed interval arithmetic into the positive semi-axis. As we can see in the Fig. 4 the largest overestimation is obtained using classical interval arithmetic. Application of directed interval arithmetic gives a significant improve of accuracy, however the expected shape of boundary can be obtained only using directed interval arithmetic mapped into the positive semi-axis.

Finally, we decide to apply proposed mapping of directed interval arithmetic into positive semi-axis for the example presented in Fig. 2. Therefore, we obtain domains defined in interval way (Fig. 5) with separate internal and external edge. From the point of view of

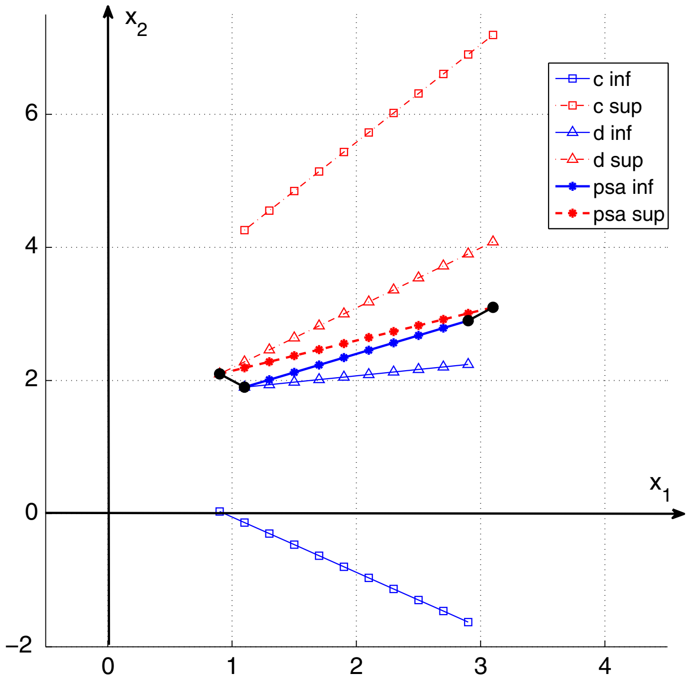


Fig. 4 Comparison between described methods on exemplary interval segment

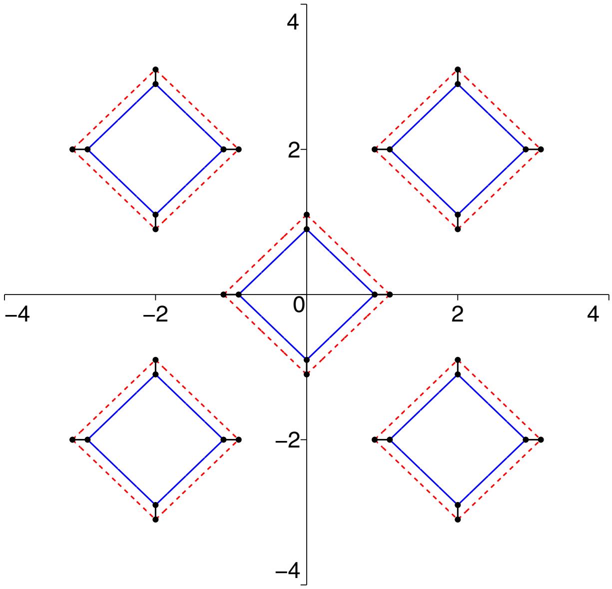


Fig. 5 The shape of interval boundary interpolated using Gauss elimination with modified directed interval arithmetic

practical modelling of boundary value problems the shapes of domains are more reliable then previously obtained. Additionally, this strategy provides the same shape of boundary regardless of the quadrant of coordinate system where corner points are defined. Moreover, we obtain exactly the same shape of boundary as expected one (Fig. 2).

7 Modelling of the smooth closed curve using interval arithmetic

7.1 Application of classical interval arithmetic and its flaws

As is presented in Sect. 5, classical interval arithmetic in interpolation of polygonal domain composed of linear segments gives worse results than directed interval arithmetic. However, directed interval arithmetic without modifications do not give reliable and unambiguous results, as well. Therefore, in this section we decide to examine once again what is the impact of classical interval arithmetic on modelling interval segments. However, we take to account example of interval cubic Bézier segments patched together producing smooth interval closed curve.

We consider interval closed curve in the shape of circle, defined by four interval cubic Bézier segments $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ (expected shape of the curve is presented in Fig. 6a). Coordinates of control points are defined using classical interval numbers, therefore each point has the form of square.

Each cubic Bézier segment ($\mathbf{S}_i, i = 0, 1, 2, 3$) is defined by 4 interval Bézier control points (presented in Fig. 6a). The endpoints of the segment ($\mathbf{P}_i, i = 0, 3, 6, 9$) are called as interpolating points, while the other points (unnumbered squares) as approximated ones. To define presented closed curve we should use 16 interval control points, however four of them are doubled at points in which segments join. Finally, we use 12 interval points only.

We are able to define closed curve in easy way using cubic B-spline curves and to enforce C^2 continuity at points in which segments join $\mathbf{P}_0, \mathbf{P}_3, \mathbf{P}_6, \mathbf{P}_9$. In practice, it requires only 4 interval de Boor control points (schematically presented in Fig. 6b and labelled

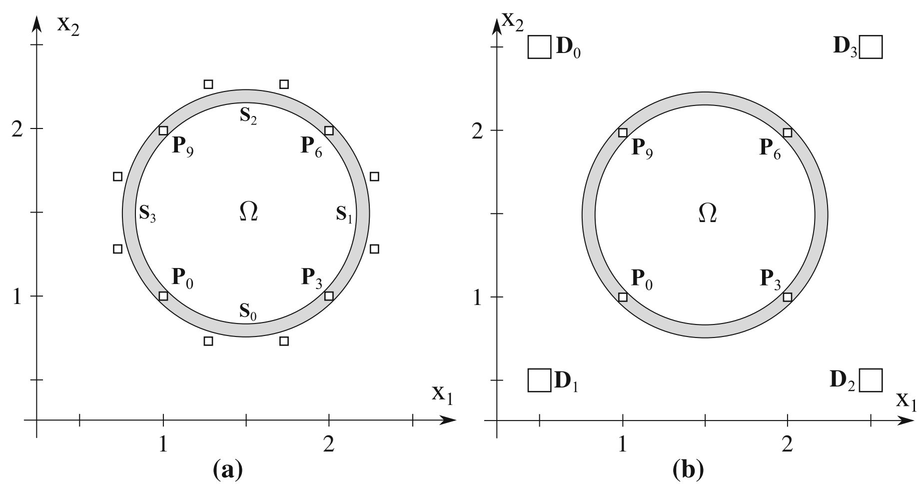


Fig. 6 Expected shape of the interval closed curve defined by **a** 12 interval Bézier control points, **b** 4 interpolating interval Bézier and 4 approximating interval de Boor control points

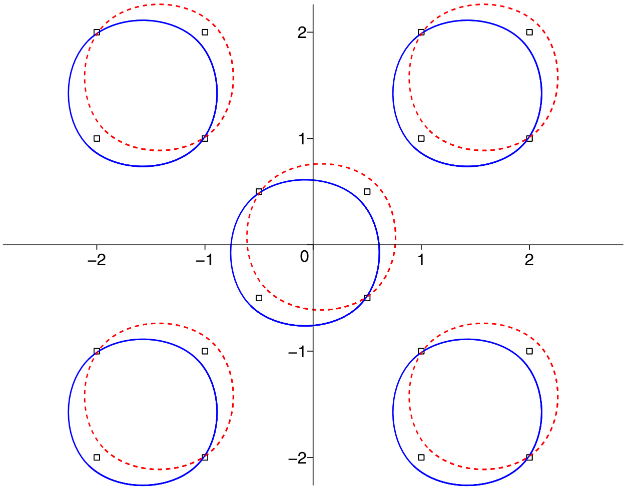


Fig. 7 Interval closed curve defined by 4 interval de Boor control points in all quadrants of Cartesian coordinate system

$\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$). Coordinates of interval de Boor control points are calculated using interval Bézier interpolating points only. For this purpose, we generalize relationships between precise Bézier and de Boor control points (Boehm 1982) and solve the following interval system of linear equations:

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{D}_0 \\ \mathbf{D}_1 \\ \mathbf{D}_2 \\ \mathbf{D}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_3 \\ \mathbf{P}_6 \\ \mathbf{P}_9 \end{bmatrix} \tag{12}$$

Generalization is related to defining Bézier and de Boor control points by interval numbers (white squares in Fig. 6). As the result of solving of interval system of equations (12) using the classical interval arithmetic we obtain interval de Boor control points (presented in Fig. 6b). The shapes of particular segments are modelled using obtained interval de Boor control points and the following formula:

$$\mathbf{S}(u) = \frac{1}{6} \left[u^3 \; u^2 \; u \; 1 \right] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{D}_0 \\ \mathbf{D}_1 \\ \mathbf{D}_2 \\ \mathbf{D}_3 \end{bmatrix} \tag{13}$$

Visualization of interval curve for infimum and supremum values of interval coordinates of de Boor control points is very interested from practical point of view. Due to the influence of interval arithmetic on solutions of system of linear equations, the curve is modelled in 4 quadrants and in the centre of Cartesian coordinate system. The visualization is presented in Fig. 7.

Shapes of the edges for the infimum and supremum of interval values of de Boor control points are independent of the quadrant of coordinate system. It is more reasonable (even with application of classical interval arithmetic) comparing to previously considered square domain. However, in previous example the matrix contained interval numbers, while now it is real number matrix. Despite this fact, edges of interval boundary intersect each other

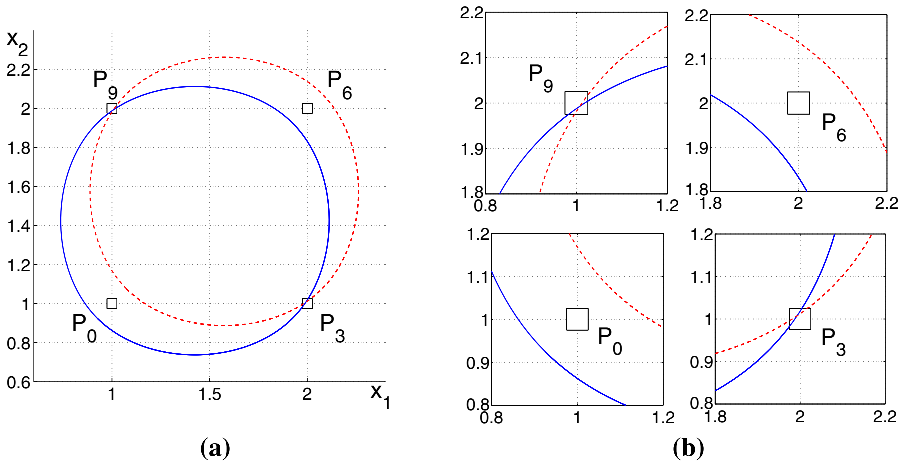


Fig. 8 **a** Visualization of interval boundary, **b** magnification at points in which adjacent segments join

in the neighbourhood of points $\mathbf{P}_3$, $\mathbf{P}_9$, therefore internal edge becomes external one. Furthermore, in the neighbourhood of points $\mathbf{P}_0$, $\mathbf{P}_6$ edges lies outside squares which represent given interpolating control points. It can be treated as a certain shift, which is connected with defining the interval points in classical way. The points are given as proper intervals, therefore infimum of interval value is smaller than supremum. However, for modelling in Cartesian coordinate system internal edge should be defined by supremum of interval value of some control points. More detailed visualization of interval boundary, particularly in the neighbourhood of interpolating interval control points is presented in Fig. 8.

Presented figures confirmed continuity of the curve at interval points in which adjacent segments join, however there is a noticeable shift of internal and external edges of interval boundary. It is difficult to explain from physical point of view. Application of these curves is troublesome, especially if we want to use the curves for modelling of uncertainly defined shape of boundary. In other words, application of classical interval arithmetic is unacceptable in case of uncertainly defined curvilinear domains, as well.

7.2 Application of directed interval arithmetic

In order to avoid the disadvantages of the above mentioned approach, we decide to use directed interval numbers. They allow to use improper intervals. Therefore, we can define the edge of interval boundary in a more intuitive way, that is internal edge of boundary is exactly inside the domain created by external one. We apply directed interval arithmetic to solve interval system of linear equations (12). We obtain interval de Boor control points with different values of intervals. All interval control points are presented in Fig. 9 as two dots joined by black line (similarly to example presented in Fig. 2).

Interval de Boor control points allow to create internal and external edge of interval boundary. It is presented for all quadrants of coordinate system in Fig. 10. Similarly to previous example, shapes of the edges for infimum and supremum of interval de Boor control points are independent of the quadrant of coordinate system. Furthermore, internal edge of boundary is exactly inside the domain created by external one (presented in Fig. 10). Therefore, this way of modelling seems to be more intuitive and reasonable than classical

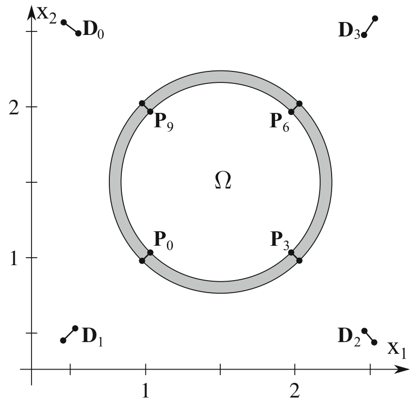


Fig. 9 De Boor control points obtained using directed interval arithmetic (presented interval shape of boundary is expected, not real)

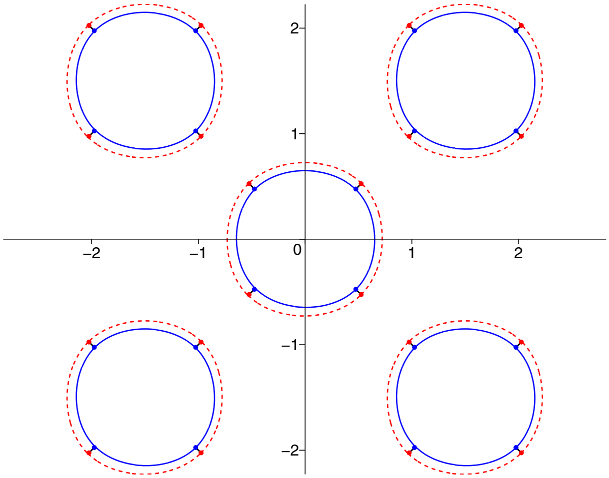


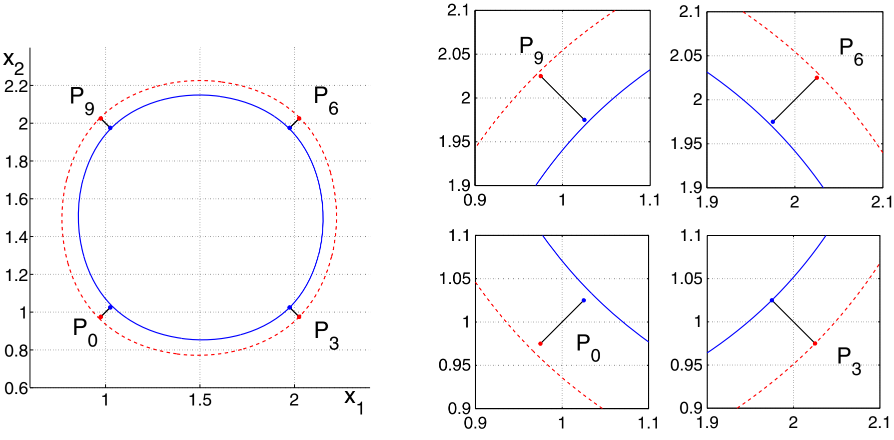
Fig. 10 Interval boundary created using 4 interval B-spline segments in all quadrants of Cartesian coordinate system

approach. More detailed visualization of interval boundary, particularly in the neighbourhood of interpolating interval control points is presented in Fig. 11.

Presented figures confirm that application of directed interval arithmetic enforced C^2 continuity for the internal and external edges. The edges do not intersect each other as in case of using classical interval arithmetic. In addition, the edges intersect points which represent interval control point $\mathbf{P}_3$. However, in other interval control points we can noted noticeable distances between the edges and the points represent control points.

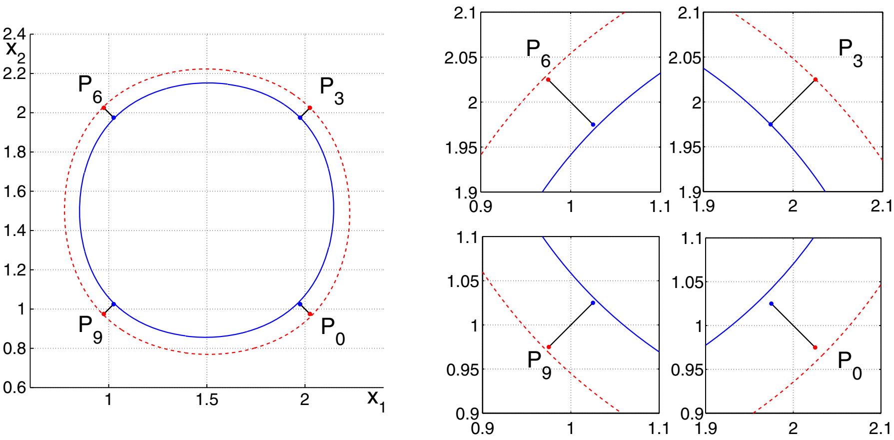
Intuitively, one would expect that the edges will intersect all interval control points, which define interval domain. Therefore, we decide to change the sequence of defining control points (presented in Fig. 12). It should not affect the shape of obtained closed curve. We can observe only the change of order of equations in the interval system of algebraic equations (12) that considering the relationship between Bézier and de Boor control points.

The change of the sequence of defining control points affects the shape of edges in the neighbourhood of interval control points. It seems to be unreliable because we should obtained the same results for identical curves regardless of the position of the first point of curve



(a) **(b)**

Fig. 11 **a** Visualization of interval boundary, **b** magnification at points in which adjacent segments join



(a) **(b)**

Fig. 12 **a** Visualization of interval boundary, **b** magnification at points of segments join (the sequence of interval control points is changed)

definition. The errors occurs due to the lack of accuracy of solving interval system of linear equations. Therefore, in the next step of our studies we apply our new strategy of modification of directed interval arithmetic.

7.3 Improving reliability of modelling using the strategy of modification of directed interval arithmetic

In order to obtain the edges of interval boundary intersecting the points representing interval control points, we decide to improve the accuracy of solving interval system of linear equations (12). Similarly to the example of modelling of polygonal domain, we map directed interval arithmetic into the positive semi-axis using (11). Application of this strategy allows to obtain the same results in all quadrants of coordinate system. Visualization of the results is presented in Fig. 13.

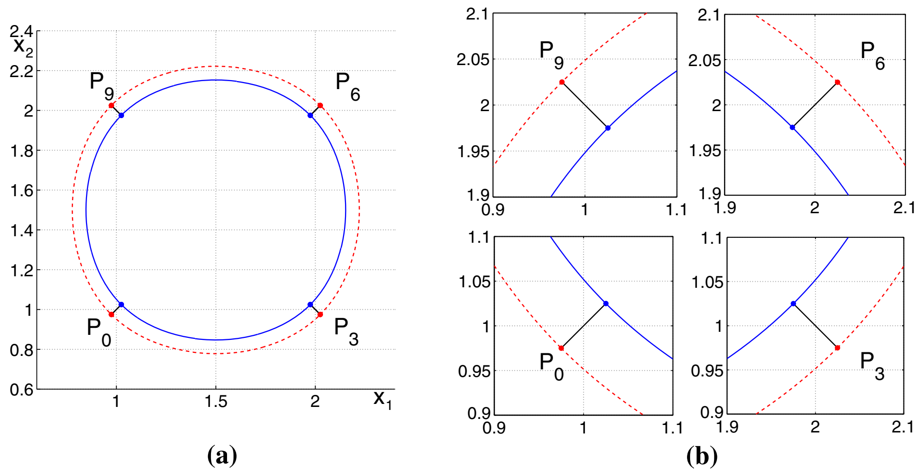
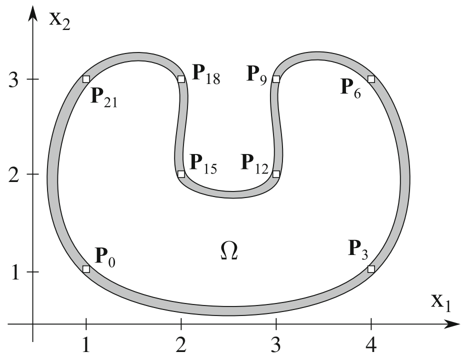


Fig. 13 **a** Visualization of interval boundary, **b** magnification at points of segments join (with applied strategy of improving the accuracy of solutions)

Fig. 14 Expected shape of interval closed curve defined by 8 interval cubic segments

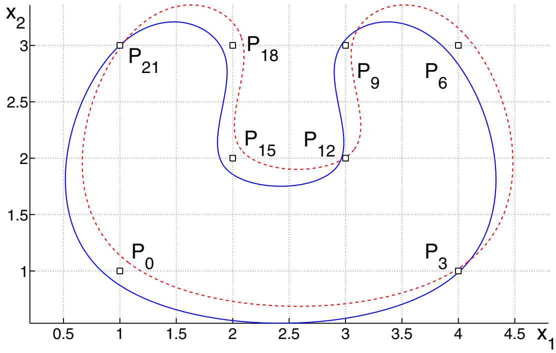


The edges of interval boundary intersect exactly all points which represent interval control points. In addition, we obtain exactly the same shape of boundary in all quadrants of coordinate system, regardless to the change of the sequence of defining control points, as well. Therefore, application of such strategy allows to obtain results reasonable from practical point of view for curvilinear interval boundary.

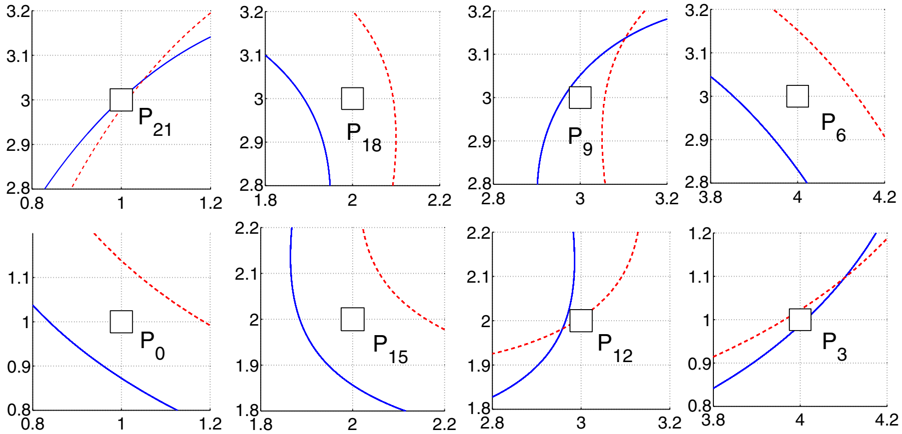
8 Verification of the strategy for more complex example

To ensure whether the described method of modelling of interval domain is effective and C^2 continuity is enforced, we consider more complex shape of interval boundary. The expected shape of the boundary is presented in Fig. 14. It is defined by 8 cubic interval Bézier segments.

To define this curve we should apply 24 interval Bézier control points. Assuming that C^2 continuity is enforced, it is enough to apply only 8 interval Bézier interpolation control points to model the domain using interval cubic B-spline curves. Interval de Boor control points are calculated similarly to previous example. However, applied interval system of linear equations is respectively greater. We solve the system by interval Gauss elimination



(a)



(b)

Fig. 15 **a** Visualization of interval boundary created by 8 interval B-spline segments, **b** magnification at points in which adjacent segments join (classical interval arithmetic approach)

method with classical interval arithmetic. Visualization of the edges of interval boundary obtained on the base of interval de Boor control points is presented in Fig. 15.

Presented figure confirms that for more complex domain application of classical interval arithmetic gives solutions unusable from practical point of view, similarly to the previous example. Presented example confirms previous conclusion, that the use of classical interval arithmetic is unacceptable from practical point of view.

Direct application of directed interval arithmetic to solve interval system of linear equations gives the same accuracy of solutions as in previous example. C^2 continuity in interval control points are enforced and the edges of interval boundary do not intersect, as well (Fig. 16).

However, edges of interval boundary do not intersect points which represent interval control points. Therefore, we decided to improve the accuracy of the solution, as in previous example. We applied the same strategy of mapping of directed interval arithmetic into the positive semi-axis (11). Therefore, we obtain interval boundary presented in Fig. 17.

Our new strategy gives accurate results even for more complex interval boundary, as well. In addition, we decide to find the edges of boundary separately for the internal and the

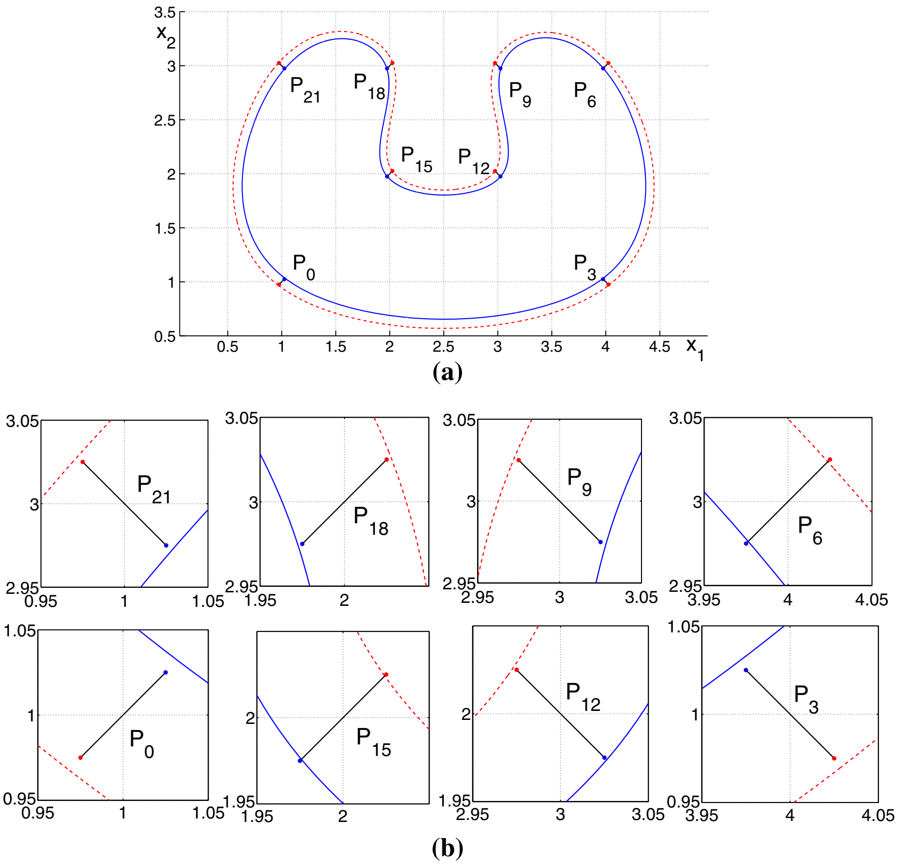
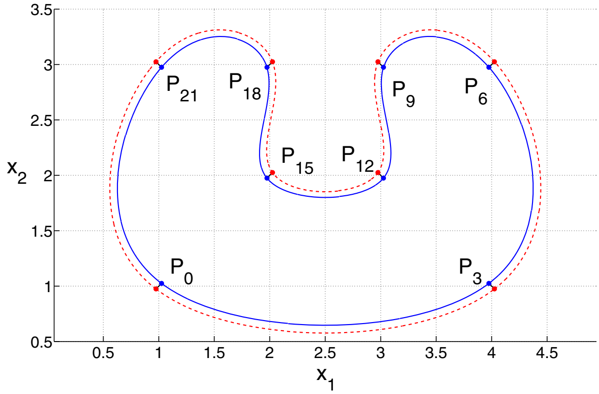


Fig. 16 **a** Visualization of interval boundary created by 8 B-spline segments, **b** magnification at points of segments join (directed interval arithmetic approach)

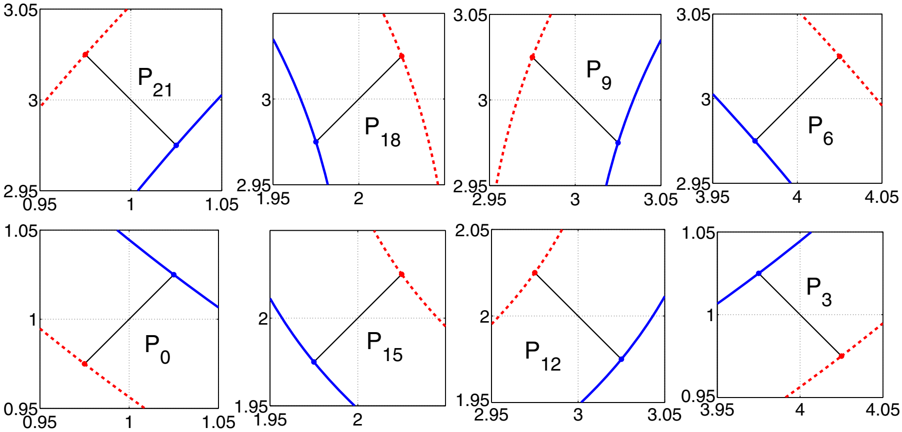
external curves without application of interval arithmetic. We solve two systems of equations and obtain two edges of boundary separately. They exactly overlapped the edges obtained in the interval way. It confirms the reliability of modelling of interval curves using our new strategy of modification of directed interval arithmetic.

9 Conclusions

The paper presents the concept of modelling interval cubic segments patched together producing interval closed curve. Closed curves can be used for modelling, among others, uncertainly defined boundary problems which can be solved by the method of PIES. Interval curves obtained in that way can also be used for modelling uncertainly defined problems related to computer graphics. In this paper, we apply interval Bézier segments to obtain interval smooth closed curves. Application of these segments is very effective because we use only two interval control points located at the ends of segment. Such location significantly simplifies the process of modelling of uncertainly defined domain. However, that way of modelling has some disadvantages.



(a)



(b)

Fig. 17 **a** Visualization of interval boundary created by 8 B-spline segments, **b** magnification at points of segments join (with applied strategy of improving the accuracy of solutions)

The main disadvantage is that at points in which adjacent segments join C^2 continuity is not automatically enforced. Such continuity of the boundary is required in many practical problems. Therefore, in order to maintain the continuity interval Bézier control points are mapped into interval de Boor control points. For this purpose, we generalize known in the literature for precise curves algebraic relations between these points. As the result of generalization, we obtain interval system of algebraic equations. To solve this system we use classical as well as directed interval arithmetic. Obtained results are different, depending on the type of arithmetic. Therefore, there was a need to carry out extensive tests to draw conclusions which arithmetic should be used for more reasonable modelling of interval curves.

In case of application of classical interval arithmetic we observe intersection of the internal and the external edges of interval boundary. The edges create two domains shifted with respect to each other. It is connected with the way of defining control points in the form of classical intervals. The internal edge of boundary in Cartesian coordinate system is not always defined by infimum of interval coordinates of control points, as well as the external edge by supremum. Such approach gives solutions which are difficult to explain from a physical point of view, especially in connection with modelling of uncertainly defined domains. Therefore, the curves defined using classical interval arithmetic are unacceptable for modelling interval domains. The conclusions is, that classical interval arithmetic is not suitable for computations related to modelling interval closed curves, which we want to use to model uncertainly defined boundary problems.

Therefore, we propose to apply directed interval arithmetic in modelling and solving uncertainly defined boundary problems. It allows to define interval control points by improper intervals. The use of directed interval arithmetic in computations is more suitable for modelling uncertainly defined curves in comparison with classical interval arithmetic, however it have some disadvantages, as well. Obtained edges of interval boundary do not intersect contrary to previous approach. Nevertheless, the edges are lying at different distances from interpolating interval Bézier control points (points in which adjacent segments join). Visualization of obtained edges of interval boundary indicates the insufficient accuracy of the calculations.

Finally, to improve accuracy of solutions, we propose the strategy of mapping directed interval arithmetic into the positive semi-axis. It allows to obtain edges of boundary which intersect the points represent interval interpolation control points. The correct results are obtained regardless of the location of the domain in coordinate system as well as for the changes in the sequence of defining interval control points, contrary to the case of direct application of directed interval arithmetic. In the authors opinion, such way of obtaining interval curves is suitable for modelling uncertain domains and definitely can be used to define uncertain boundary problems solving by interval version of PIES method.

References

Boehm W (1982) On cubics: a survey. Comput Graph Image Process 19:201–226

Bołtuć A, Zieniuk E (2011) Modeling domains using Bézier surfaces in plane boundary problems defined by the Navier–Lamé equation with body forces. Eng Anal Bound Elem 35:1116–1122

Chen F, Deng L (2004) Interval implicitization of rational curves. Comput Aided Geom Des 21:401–415

Chen F, Lou W (2000) Degree reduction of interval Bézier curves. Comput Aided Des 32:571–582

Csebfalvi B (2010) An evaluation of prefiltered B-spline reconstruction for quasi-interpolation on the body-centered cubic lattice. IEEE Trans Vis Comput Graph 16:499–512

Dimitrova N, Markov SM, Popova E (1992) Extended interval arithmetics: new results and applications. In: Atanassova L, Herzrberger J (eds) Computer arithmetic and enclosure methods. North-Holland, Amsterdam, pp 225–232

Galvez A, Iglesias A (2013) Firefly algorithm for polynomial Bézier surface parameterization. J Appl Math 2013, Article Number: 237984

Goldsztejn A, Chambert G (2007) A generalized interval LU decomposition for the solution of interval linear systems. Lect Notes Comput Sci 4310:312–319

Gonzalez-Hidalgo M, Jaume-i-Capo A, Mir A, Nicolau-Bestard G (2013) A tool for analytical simulation of B-splines surface deformation. Comput Aided Des 45:168–179

Hanniel I, Wein R (2009) An exact, complete and efficient computation of arrangements of Bézier curves. IEEE Trans Autom Sci Eng 6:399–408

Hansen ER (1997) Sharpness in interval computations. Reliab Comput 3:17–29

Hansen E, Walster GW (2004) Global optimization using interval analysis. Marcell Dekker, New York
Hao MZ, Sun YW (2013) A FEM model for simulating temperature field in coaxial laser cladding of TI6AL4V alloy using an inverse modeling approach. Int J Heat Mass Transf 64:352–360
Hayes B (2003) A Lucid interval. Am Scientist 91:484–488
Ismail O (2014) Degree elevation of interval B-spline curves. Int J Video Image Process Netw Secur 14(2):9–14
Ismail O (2014) Degree reduction of interval B-spline curves. Int J Video Image Process Netw Secur 14(3):1–5
Kużelewski A, Zieniuk E (2015) GPU-based acceleration of computations in elasticity problems solving by parametric integral equations system. Adv Eng Softw 79:27–35
Kużelewski A, Zieniuk E (2015) Acceleration of integration in parametric integral equations system using CUDA. Comput Struct 152:113–124
Lin HW, Liu LG, Wang GJ (2002) Boundary evaluation for interval Bézier curve. Comput Aided Des 34:637–646
Markov S (1995) On directed interval arithmetic and its applications. J Univ Comput Sci 1:510–521
Markov S (1999) An iterative method for algebraic solution to interval equations. Appl Numer Math 30:225–239
Martin R, Shou H, Voiculescu I, Bowyer A, Wang G (2002) Comparison of interval methods for plotting algebraic curves. Comput Aided Geom Des 19:553–587
Moore RE, Kearfott RB, Cloud MJ (2009) Introduction to interval analysis. SIAM, Philadelphia
Moore RE (1966) Interval analysis. Prentice-Hall, Englewood Cliffs
Muhanna RL, Zhang H, Mullen RL (2007) Combined axial and bending stiffness in interval finite-element methods. J Struct Eng-ASCE 133:1700–1709
Neumaier A (1990) Interval methods for systems of equations. Cambridge University Press, Cambridge
Piasecka-Belkhayat A (2011) Interval boundary element method for transient diffusion problem in two-layered domain. J Theor Appl Mech 49:265–276
Sederberg TW, Farouki RT (1992) Approximation by interval Bézier curves. IEEE Comput Graph Appl 12:87–94
Shary SP (2012) Solvability of interval linear equations and data analysis under uncertainty. Autom Remote Control 73:310–322
Stahl V (1997) Sufficient condition for non-overestimation in interval arithmetic. Computing 59:349–363
Tuohy ST, Maekawa T, Shen G, Patrikalakis NM (1997) Approximation of measured data with interval B-spline. Comput-Aided Des 29:781–799
Zalewski BF, Mullen RL (2009) Local discretization error bounds using interval boundary element method. Int J Numer Meth Eng 78:403–428
Zieniuk E (2003) Bézier curves in the modification of boundary integral equations (BIE) for potential boundary-values problems. Int J Solids Struct 40:2301–2320
Zieniuk E, Bołtuć A, Kużelewski A (2007) Algorithms of identification of multi-connected boundary geometry and material parameters in problems described by Navier-Lamé Equation using the PIES. In: Pejas J, Saeed K (eds) Advances in information processing and protection. Springer, New York, pp 409–418
Zieniuk E, Kapturczak M, Kużelewski A (2016) Solving interval systems of equations obtained during the numerical solution of boundary value problems. Comput Appl Math **(online)**. doi:10.1007/s40314-014-0209-9
Zieniuk E, Kużelewski A (2015) Concept of the interval modelling the boundary shape using interval Bézier curves in boundary problems solved by PIES, In: 12th international conference of numerical analysis and applied mathematics ICNAAM 2014. AIP conf. proc., vol 1648, p 590002. doi:10.1063/1.4912829
Zieniuk E, Szerszeń K (2013) Triangular Bézier surface patches in modeling shape of boundary geometry for potential problems in 3D. Eng Comput 29:517–527
Zieniuk E, Szerszeń K (2014) The PIES for solving 3D potential problems with domains bounded by rectangular Bézier patches. Eng Comput 31:791–809